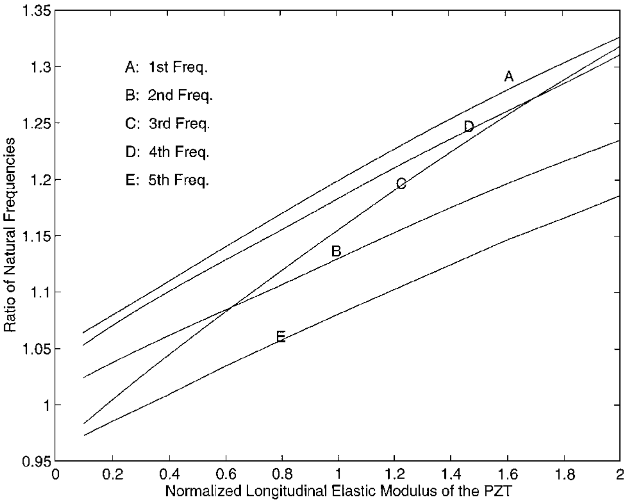

Fig. 3 Frequency of the plate with two PZT layers/frequency of the plate without the PZT layers vs C_{11} for the PZT layer/C_{11} for the 0-deg lamina.

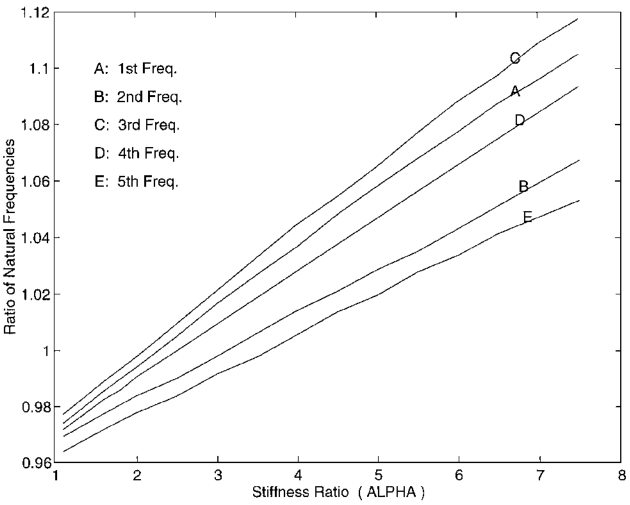

Fig. 4 Frequency of the plate with two PZT layers/frequency of the plate without the PZT layers vs α.

with an increase in the mass density of the PZT layer. The five curves are nearly parallel to each other, signifying that the rate of change of frequency with respect to the mass density of the PZT layer is the same for all five frequencies. As for the variation in the thickness of the PZT layer, the curve for the fourth frequency lies between those for the first and the third ones and that for the third frequency lies between those for the third and fifth ones.

Because the substrate layer has been modeled as orthotropic and the PZT layer as transversely isotropic, it is not clear how to vary the stiffness of the PZT layer relative to that of the graphite/epoxy substrate. In an attempt to decipher the effect of the material stiffness only, the mass density and the thickness of the PZT layer are first set equal to that of the graphite/epoxy lamina, and only C_{11} for the PZT layer is varied. Smart structure's response is also affected by values of other components of $\boldsymbol{C}$ for the PZT. As shown in Fig. 3, all five lowest natural frequencies of the composite structure increase monotonically with an increase in the value of C_{11} for the PZT. In the second study of the effect of the PZT stiffness on the natural frequencies of the plate, the thickness of the PZT layer is taken to be equal to $\frac{1}{10}$th of the thickness of the substrate layer, which is more likely to occur in a physical situation, mass density of the PZT layer equal to 7500 kg/m^3, $\boldsymbol{C}^{\text{PZT}} = \alpha \boldsymbol{C}^{\text{substrate}}$, and α is varied. Of course, it may be difficult to manufacture a PZT with such mechanical properties. Figure 4 illustrates the effect of such a change in the elastic moduli of the PZT on the five lowest natural frequencies of the composite plate. A sevenfold increase in the stiffness of the PZT increases the first five natural frequencies by at most 10%.

Conclusions

For the simply supported composite plate studied herein, it is found that an increase in the thickness of the PZT layer or its mass density monotonically decreases each one of the five lowest frequencies and an increase in the stiffness of the PZT layer relative to that of the lamina increases these frequencies. However, the relative change in these frequencies is not necessarily the same.

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Bifurcation Buckling Analysis of Delaminated Composites Using Global–Local Approach

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I. Introduction

THE buckling and postbuckling analyses of laminated composites with multiple delaminations are important because a low-speed impact generates multiple delaminations through the laminates. However, an analytical solution approach is only possible for simple delamination zone geometry and boundary conditions because the buckling and postbuckling problems are complicated.[1] Therefore, the finite element method is preferable to analyze the buckling behavior of multiply delaminated composites for arbitrary layup configurations, boundary conditions, loading conditions, and delaminated zone shapes.

Lee et al. analyzed the bifurcation buckling[2] and postbuckling behavior[3] of a laminated composite beam with layerwise finite elements proposed by Reddy.[4] However, for multiple-delaminated composites with layer-dependent finite elements, a large number of degrees of freedom (DOF) is required through the thickness because of the geometric complexity of the delaminated zones. Thus, much computer memory and computing time are required. Furthermore, the stability analysis of laminated structures is a nonlinear problem

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because contacts between the interfaces of the delaminated zones should be considered. In addition, iterative solution techniques are needed to solve the postbuckling problem with geometric nonlinearity. This complicated problem can be efficiently analyzed by a global–local approach.

The global–local approach is frequently used in the computational structural analysis. However, until now, global–local approaches have not been applied to buckling analysis of composite laminates with delaminations. Therefore, the objective of the present study is the analysis of buckling of composite laminates with multiple delaminations using a global–local approach. Thus, computer resources can be saved without loss of accuracy.

Layerwise elements are used in the local delaminated zone because they accurately describe the geometric deformations of the delaminated zone and are still computationally less expensive than three-dimensional elements. In the undelaminated part, the first-order shear elements are used because the deformation patterns are less complicated than those of the delaminated zone. A transition element is developed to connect the global undelaminated zone and local delaminated zone. This transition element can be easily implemented by applying an existing postprocess method.[5,6] The present transition element connects layerwise elements and first-order shear elements.

Consequently, an efficient three-dimensional delamination buckling analysis tool, which works on a personal computer or a workstation, can be developed.

II. Formulations

A. Problem Statement

A beam-plate problem with multiple delaminations is considered. Delamination cracks can be placed at any position along the in-plane direction. The number of delaminations and the size of each delamination are chosen arbitrarily. It is assumed that the crack lengths of the delaminated zones are not increased when loading is applied. Unilateral contacts between the interfaces of delaminated zones are not considered.

B. Layerwise Model and First-Order Model

An N-layer, fiber-reinforced, composite beam-plate with multiple, through-the-width delaminations is considered.

The resulting in-plane and out-of-plane displacements of the layerwise model, U_α and U_3, at a generic point x, y, z in the laminate are assumed to be of the following form.[2] Subscript α can be 1 or 2:

$$U_\alpha(x,y,z) = u_\alpha(x,y) + \sum^{N} \phi^j(z) u^j_\alpha(x,y) + \underline{\sum^{D} \delta(z) \bar{u}^i_\alpha(x,y)}$$

$$U_3(x,y,z) = w(x,y) + \underline{\sum^{D} \delta(z) \bar{w}^i(x,y)} \tag{1}$$

where N is the total number of layers and D is the total number of delaminations. The underlined parts denote the delamination kinematics. The terms $\bar{u}^i_\alpha$ and $\bar{w}^i$ represent possible jumps in the slipping and the opening displacements, respectively, at the $L(i)$th delaminated interface. $L(i)$ is the location of an interface where ith delamination lies, and $\phi^j(z)$ is a linear Lagrangian interpolation function through the thickness of the laminates. The jump function $\delta(z)$ can be represented as Heaviside unit step function $H(z)$,

$$\delta(z) = H(z - z_{L(i)}) - H(-z - z_{L(i)}) \tag{2}$$

The displacement field of the first-order shear deformation plate theory (fopt) is given as follows:

$$U_\alpha(x,y,z) = u_\alpha(x,y) + \psi_\alpha(x,y)z, \qquad U_3(x,y,z) = w(x,y) \tag{3}$$

where u_α and w are the midplane stretching and the out-of-plane deflection, respectively, and ψ_α is the angle of rotation at the midplane.

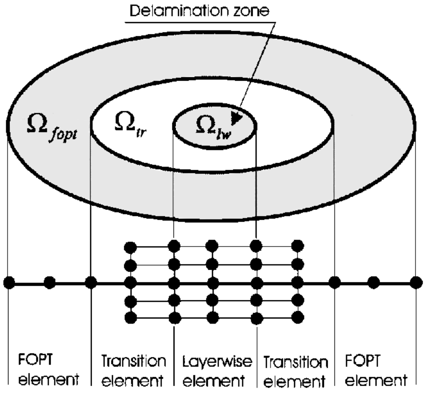


Fig. 1 Definition of the transition domain.

C. Matching Relationship Between Layerwise and First-Order Models

Consider a domain Ω modeled as three independently discretized subdomains, Ω_{lw}, Ω_{tr}, and Ω_{fopt}, as shown in Fig. 1. The subscripts lw, tr, and fopt denote the layerwise domain, the transition domain, and the fopt domain, respectively.

The total potential energy of the system is the sum of the potential energies of subdomains Ω_{lw}, Ω_{fopt}, and Ω_{tr},

$$\begin{aligned}\Pi_\Omega &= \Pi_{\Omega_{\text{lw}}} + \Pi_{\Omega_{\text{fopt}}} + \Pi_{\Omega_{\text{tr}}} \\ &= \tfrac{1}{2}\boldsymbol{U}^T_{\text{lw}}\boldsymbol{K}_{\text{lw}}\boldsymbol{U}_{\text{lw}} + \tfrac{1}{2}\boldsymbol{U}^T_{\text{fopt}}\boldsymbol{K}_{\text{fopt}}\boldsymbol{U}_{\text{fopt}} - \boldsymbol{U}^T_{\text{lw}}\boldsymbol{F}_{\text{lw}} \\ &\quad - \boldsymbol{U}^T_{\text{fopt}}\boldsymbol{F}_{\text{fopt}} + \tfrac{1}{2}\boldsymbol{U}^T_{\text{tr}}\boldsymbol{K}_{\text{tr}}\boldsymbol{U}_{\text{tr}} - \boldsymbol{U}^T_{\text{tr}}\boldsymbol{F}_{\text{tr}}\end{aligned} \tag{4}$$

where

$$\boldsymbol{U}_{\text{lw}} = \left\{u_{\alpha 1}, w_1, u^j_{\alpha 1}, \bar{u}^i_{\alpha 1}, \bar{w}^i_{\alpha 1}, \ldots, u_{\alpha n}, w_n, u^j_{\alpha n}\bar{u}^i_{\alpha n}, \bar{w}^i_{\alpha n}\right\}^T$$
$$(j = 1, \ldots, N), \qquad (i = 1, \ldots, D) \tag{5}$$

$$\begin{aligned}\boldsymbol{U}_{\text{tr}} = \{&u_{\alpha 1}, w_1, \psi_{\alpha 1}, u_{\alpha 2}, w_2, u^j_{\alpha 2}, \bar{u}^i_{\alpha 2}, \bar{w}^i_{\alpha 2}, \ldots, \\ &u_{\alpha n}, w_n, u^j_{\alpha n}\bar{u}^i_{\alpha n}, \bar{w}^i_{\alpha n}\}^T\end{aligned} \tag{6}$$

$$\boldsymbol{U}_{\text{fopt}} = \left\{u_{\alpha 1}, w_1, \psi_{\alpha 1}, \ldots, u_{\alpha n}, w_n, \psi_{\alpha n}\right\}^T \tag{7}$$

$\boldsymbol{K}$, $\boldsymbol{U}$, and $\boldsymbol{F}$ refer, respectively, to the stiffness matrices, the generalized displacement, and the load vector for each domain; and n is the number of nodes for each domain.

The transition stiffness matrix $\boldsymbol{K}_{\text{tr}}$ is calculated from

$$\boldsymbol{K}_{\text{tr}} = \boldsymbol{T}^T \boldsymbol{K}_{\text{lw}} \boldsymbol{T} \tag{8}$$

The transformation matrix $\boldsymbol{T}$ is calculated from transverse shear strain relationship between the first-order model and the layerwise model,

$$w_{,\alpha} + \sum^{N} \phi^j_{,3} u^j_\alpha = c^k_\alpha (w_{,\alpha} + \psi_\alpha)^{\text{fopt}} \tag{9}$$

where the matching factor c^k_α is constant for each layer. The determination procedure of the matching factor is omitted here because of limited space. It can be found in detail in Ref. 7.

III. Implementation of the Transition Element

In the finite element model, the generalized displacements of the layerwise model (u_α, w, u^j_α, $\bar{u}^i_\alpha$, $\bar{w}^i_\alpha$) and the first-order shear model (u_α, w, ψ_α) are expressed over each element as a linear combination of one-dimensional quadratic Lagrangian interpolation functions N_l as follows:

$$(u_\alpha, w, u^j_\alpha, \bar{u}^i_\alpha, \bar{w}^i_\alpha) = \sum^{m} (u_{\alpha l}, w_l, u^j_{\alpha l}, \bar{u}^i_{\alpha l}, \bar{w}^i_{\alpha l}) N_l \tag{10}$$

$$(u_\alpha, w, \psi_\alpha) = \sum^{m} (u_{\alpha l}, w_l, \psi_{\alpha l}) N_l \tag{11}$$

where m is the number of nodes per element.

Using Eqs. (10) and (11) in the transvere shear strain relationship between layerwise and first-order shear theory [Eq. (9)], we obtain

$$T^e_{\text{tr}} U^e_{\text{tr}} = T^e_{\text{lw}} U^e_{\text{lw}} \tag{12}$$

It is assumed that the out-of-plane deflection w of layerwise theory is the same as that of fopt. From Eqs. (10–12), the element transformation matrix T^e is calculated as follows:

$$\begin{bmatrix} I & \{0\} \\ T^e_{\text{fopt}} & \end{bmatrix} U^e_{\text{tr}} = \begin{bmatrix} I & [0] \\ T^e_{\text{lw}} & \end{bmatrix} U^e_{\text{lw}} \tag{13}$$

$$U^e_{\text{lw}} = T^e U^e_{\text{tr}} \tag{14}$$

where T^e_{fopt} is an $N \times (m+1)$ matrix, T^e_{lw} is an $N \times (m+N)$ matrix, and I is an $m \times m$ identity matrix. Here $\{0\}$ is $m \times 1$ null vector and $[0]$ is $m \times N$ null matrix. The other components of U^e_{tr} are the same as those of U^e_{lw}.

The finite element model of the buckling problem can be obtained as

$$([K] - \lambda[S])\{u\} = \{0\} \tag{15}$$

where $[K]$, λ, $[S]$, and $\{u\}$ are the stiffness matrix, the eigenvalues, the geometric stiffness matrix, and the eigenvector of nodal displacements corresponding to an eigenvalue, respectively.

IV. Numerical Results and Discussion

To examine the accuracy of the proposed method, two cases of delaminated composite beam-plates under axial compression were considered: 1) clamped beam-plate and 2) simply supported beam-plate. The configuration for these boundary conditions is given in Fig. 2.

A. Clamped Beam-Plate

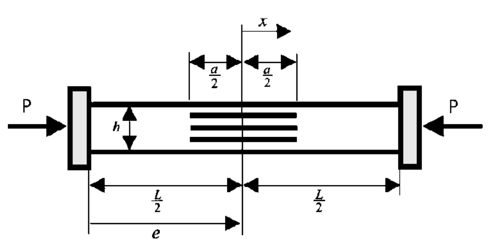
The material used in the numerical example is T300/5208 graphite/epoxy composite whose properties are as follows: $E_{11} = 181$ GPa, $E_{22} = 10.3$ GPa, $G_{12} = 7.7$ GPa, and $\nu_{12} = 0.28$, where E_{11} is Young's modulus in the fiber direction, E_{22} is the Young's modulus in the transverse direction, G_{12} is the shear modulus, and ν_{12} is Poisson's ratio.

The clamped boundary conditions for symmetric and antisymmetric modes are given in Ref. 2.

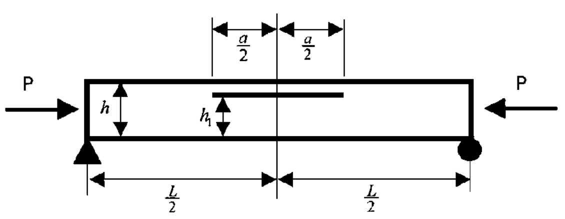
First, a specially orthotropic composite laminate containing one centrally located midplane delamination is considered. The results of this example are reproduced from Ref. 2. Nondimensional buckling loads for the three distinguishable buckling modes of a laminate with a single delamination are shown in Figs. 3a and 3b for $(h/L) = \frac{1}{400}$ and $\frac{1}{10}$, respectively. In all of the figures, LW and LF denote the full layerwise model and the present model, which uses the layerwise model (L) in the local zone and the first-order model (F) in the global zone, respectively. Even for the moderately thick plate case, the present method gives good results for all these modes.

Next, analysis is performed for a beam-plate with seven delaminations and each ply having the same thickness. The thickness-to-span ratio is assumed to be $\frac{1}{10}$. The nondimensional buckling load is compared in Fig. 4 with the results of layerwise theory. Again, the present method shows good agreements with the layerwise theory.

The effect of delamination location along the axial direction of the composite on the lowest buckling load for one, three, and seven delaminations is shown in Fig. 5. Each ply has the same thickness and the same delamination size $(a/L = 0.2)$. The thickness-to-span ratio is assumed to be $\frac{1}{50}$. In a single delamination at its midplane, the new results are compared with those of the layerwise theory. The lowest buckling mode is decreasingly sensitive to axial delamination position as the number of through-the-thickness delaminations increase. The buckling load is the highest when the delamination lies at the center of the composite. It is found that the effect of the number of through-the-thickness delaminations is similar to the effect of the delamination size of Ref. 2. In this example, the total numbers of DOF for the layerwise and present models are given in Fig. 5. The present model requires a smaller number of DOF than the

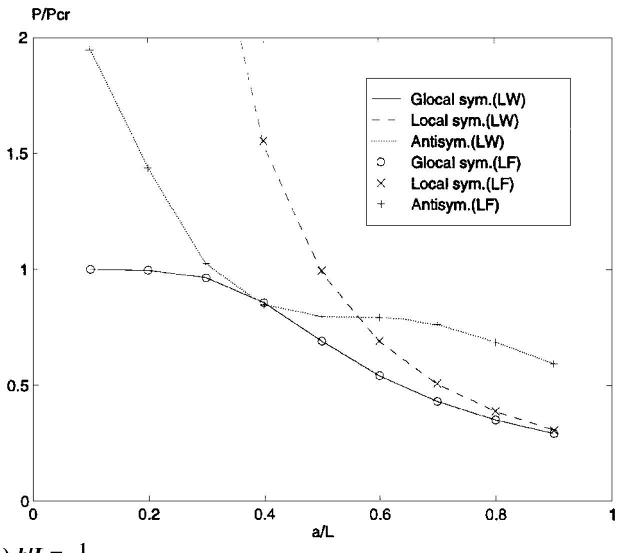


a) Geometry of a clamped composite with delaminations

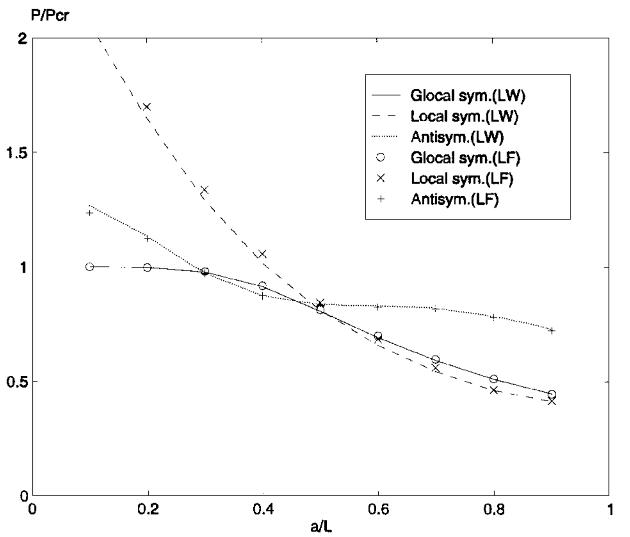


b) Geometry of a simply supported composite with one delamination

Fig. 2 Configuration of composite with delaminations.



a) $h/L = \frac{1}{400}$



b) $h/L = \frac{1}{10}$

Fig. 3 Nondimensionalized buckling load vs delamination length.

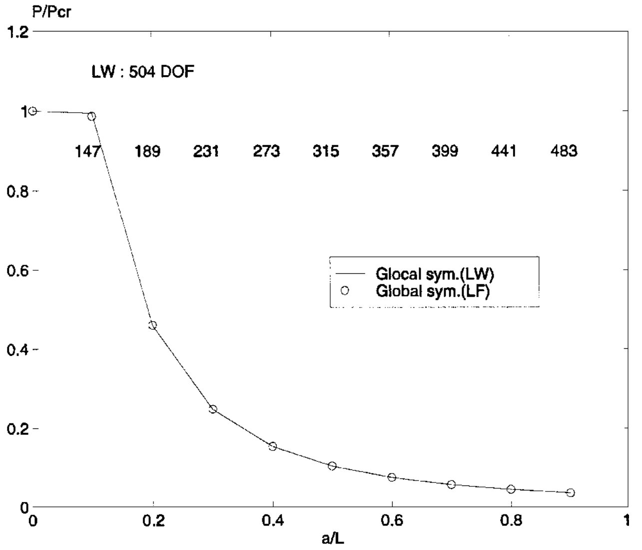


Fig. 4 Nondimentionalized buckling load for seven delaminations.

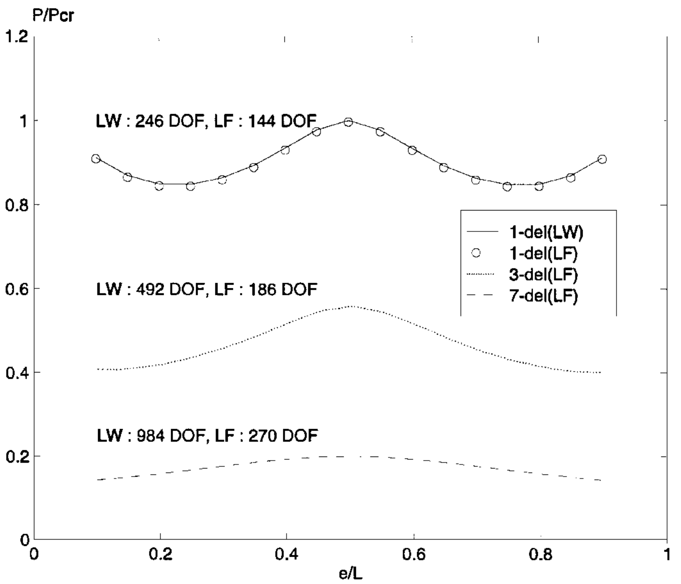


Fig. 5 Effect of axial location of one, three, and seven delaminations for *a*/*L* = 0.2.

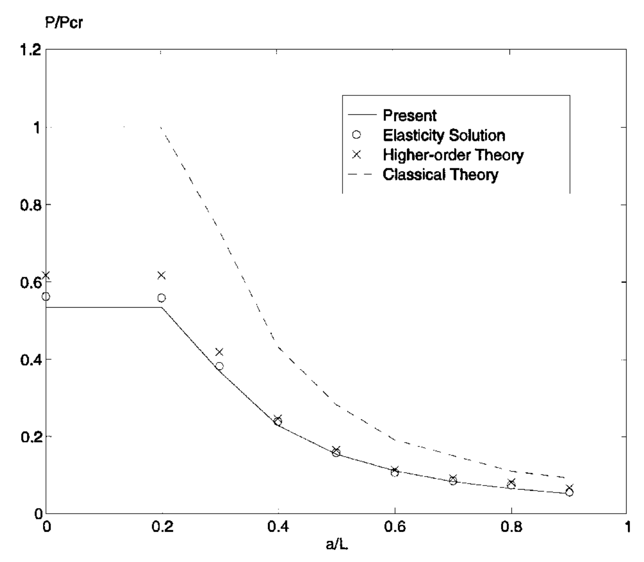


Fig. 6 Nondimensionalized buckling load vs delamination length for $[0_5/90_5/0_5]_s$ (h_1/h = 0.9).

layerwise model, especially as the number of through-the-thickness delamination increases.

B. Simply Supported Beam-Plate

The elasticity solution for delamination buckling of plate was proposed by Gu and Chattopadhyay.[8] The material used in this example is as follows: $E_L = 25 \times 10^6$ psi, $E_T = 1 \times 10^6$ psi, $G_{LT} = 0.5 \times 10^6$ psi, $G_{TT} = 0.2 \times 10^6$ psi, and $\nu_{LT} = \nu_{TT} = 0.25$, where L denotes the fiber direction and T denotes the direction perpendicular to the fiber.

The plate is thick ($L/h = 10$), and the delaminated layer is relatively thin ($h_1/h = 0.9$). Figure 6 presents the critical loads normalized by the value calculated by the classical laminated theory (CLT). The results obtained for $[0_5/90_{10}/0_5]$ composite laminates are compared with those of CLT, the higher-order theory,[9] and the elasticity solution.[8] The results of the present method show good correlation with the elasticity solutions over the entire range of delamination. Because the composite plate used in this example is thick ($L/h = 10$), the present global–local method is expected to enable an accurate evaluation of transverse shear effects and realistic prediction of delamination buckling behavior.

V. Conclusions

An efficient numerical model was developed to study the compressive stability of delaminated composites. A global–local approach for a bifurcation buckling problem with multiple delaminations was presented. A novel transition element between layerwise element region and first-order shear deformation zone was proposed by utilizing a displacement field matching method. The transition element is easy to implement, and it performs well even for thick composites when proper shear correction factors in the first-order model are chosen.

The present global–local method determines the number of DOF of undelaminated zone independently of both the number of layers and the number of delaminations. Therefore, it reduces the global DOF but still accurately models the delaminated zone. Furthermore, this method can be applied to problems with many layers and multiple delaminations.

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